2901. Splitting the fraction up, we have a pair of infinite geometric series:

$$\sum_{n=1}^{\infty}\left(\tfrac{2}{5}\right)^n+\sum_{n=1}^{\infty}\left(\tfrac{3}{5}\right)^n.$$

These converge, as each has $|r|<1$. So, we can use $S_\infty=\frac{a}{1-r}$, giving

$$S=\frac{\frac{2}{5}}{1-\frac{2}{5}}+\frac{\frac{3}{5}}{1-\frac{3}{5}}=\frac{13}{6}.$$

2902. (a) For SPs,

$$\begin{aligned}&\tfrac{1}{3}x^{-\frac{2}{3}}-\tfrac{1}{3}x^{-\frac{4}{3}}=0\\ \Longrightarrow\ &x=\pm 1.\end{aligned}$$

The second derivative is

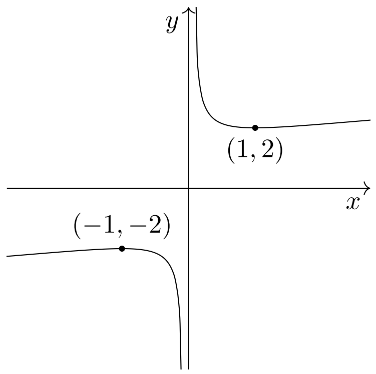
$$-\tfrac{2}{9}x^{-\frac{5}{3}}+\tfrac{4}{9}x^{-\frac{7}{3}}.$$

This has value $\pm 2/9$ at the SPs. Hence, $(1,2)$ is a local min and $(-1,-2)$ is a local max.

(b) At $x=0$, the curve is undefined. At $y=0$, we have $x^{\frac{2}{3}}=-1$, which has no real roots, since $x^2\geq 0$. So, there are no axis intercepts.

(c) There is an asymptote at $x=0$: the $x^{-\frac{1}{3}}$ term tends to $\pm\infty$ as $x\to 0^{\pm}$.

(d) As $x\to\pm\infty$, the second term tends to zero, so the curve approaches $y=x^{\frac{1}{3}}$. Putting the above information together, the behaviour is



2903. Assume, for a contradiction, that a polynomial of degree $n\geq 1$ exists which is equivalent to the given algebraic fraction:

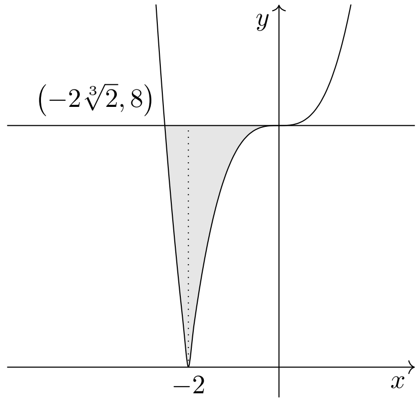
$$\mathrm{f}(x)\equiv\frac{\text{NUM}}{\text{DEN}}.$$

Both numerator and denominator have degree 3. Multiplying up by the denominator, this gives a polynomial of degree $n+3\geq 4$ equivalent to a polynomial of degree 3:

$$\underbrace{\mathrm{f}(x)\times\text{NUM}}_{\text{degree n+3}}\equiv\underbrace{\text{DEN}}_{\text{degree 3}}$$

This is a contradiction. So, there is no polynomial in x equivalent to the algebraic fraction.

2904. The first curve is $y=x^3$, translated in the positive y direction by 8 units, then with a mod function applied to it. It intersects with the line $y=8$ at $x=0$ and $x=-2\sqrt[3]{2}$.



So, the area of the shaded region is given by two integrals, one representing the area to each side of the dotted line above:

$$\begin{aligned}A&=\int_{-2\sqrt[3]{2}}^{-2}8+(x^3+8)\,dx+\int_{-2}^{0}8-(x^3+8)\,dx\\&=\left[\tfrac{1}{4}x^4+16x\right]_{-2\sqrt[3]{2}}^{-2}+\left[-\tfrac{1}{4}x^4\right]_{-2}^{0}\\&=\left(24\sqrt[3]{2}-28\right)+\left(4\right)\\&=24\left(\sqrt[3]{2}-1\right),\text{ as required.}\end{aligned}$$

2905. (a) This is a one-tailed test. We assume that the population has distribution $X\sim N(\mu,5.11^2)$, where μ is the population mean usage in kWh. The hypotheses are:

$$\begin{aligned}H_0&:\mu=11.6\\H_1&:\mu<11.6.\end{aligned}$$

(b) Assuming H_0, the mean of a sample of 49 is distributed as follows:

$$\bar{X}\sim N\left(11.6,\tfrac{5.11^2}{49}\right).$$

Solving $\mathbb{P}(\bar{X}<k)=0.02$ gives $k=10.10$. So, the critical region for the sample mean is $(\infty,10.10)$ kWh.

(c) The sample statistic is $411.4/49=8.396$. This lies in the critical region, so there is sufficient evidence, at the 2% level, to reject H_0. It does seem that usage is now below the historical value of 11.6 kWh.

(d) It is extremely unlikely that usage follows a normal distribution. To assume that any real population follows a normal distribution, even approximately, is likely to be a mistake. More often than not, such assumptions are made for the purposes of enabling academic study (without awareness of this fact), and not for the drawing of useful conclusions.

In this context, usage will almost certainly be positively skewed, as 0 kWh is a hard lower bound, while there are likely to be extreme values at the upper end of the distribution.

— Nota Bene —

Be careful about *applying* mathematics. It is a wonderful sandpit, but a terrible philosophy. Maths is one of the finest tools for training the mind; however, this doesn't mean the *world* is mathematical. Indeed, the living world, and the human world in particular, really isn't mathematical at all. For the making of human decisions (and under this umbrella I include many decisions that are currently taken based on statistical analysis), the heart is by far the better judge.

2906. The intersections are at

$$\begin{aligned} y^2 &= 8 - y^2 \\ \implies y &= \pm 2. \end{aligned}$$

We can then integrate the x distance between the curves. For $y \in (-2, 2)$, $8 - y^2 > y^2$, so the area is given by

$$A = \int_{-2}^{2} 8 - 2y^2 \, dy.$$

Using a definite integrator, this is $\frac{64}{3}$.

2907. There is no friction and the rope is light, so there can be no difference in the tension either side of the pulley. Hence, irrespective of which monkey is actually doing the climbing, the tension exerted by the rope must be the same on each. This is NIII, transmitted by the rope. The force diagrams are identical (tension upwards, weight downwards), so, by NII, both monkeys rise symmetrically.

— Nota Bene —

This is the same fact, *mutatis mutandis*, as if two astronauts are floating in space and one gives the other a push. It doesn't matter which gives the push, both move away with the same acceleration and thus the same speed.

2908. For the LHS to be zero, one of the factors must be zero. The square on the first factor has no effect; there are roots $x = 2, -3$. The second factor gives $x^2 = 2, -3$, so $x = \pm\sqrt{2}$. Hence, the equation has four roots.

2909. The integral of $\frac{1}{x}$ is $\ln|x|$. We can ignore the mod signs, as all of the limits are positive. So, the LHS is $\ln ab - \ln 1$, which is $\ln ab$. Likewise, the RHS is $\ln a + \ln b$. The equation of these is a law of logarithms: $\ln ab \equiv \ln a + \ln b$. □

2910. For a counterexample, we can take any function (other than the identity function $f(x) = x$) which is self-inverse. So, consider $f(x) = -x$, for which $f^2(x) = x$. For example, $x_0 = 2$ is a fixed point of f^2, but not of f.

2911. (a) Substituting $t = 0$, we get $\ln(2R) = 0$, which is $R = \frac{1}{2}$. This corresponds to 5000 cells per millilitre of blood.

(b) Solving $\frac{1}{2}e^{2t-t^2} = \frac{1}{2}$, we require $2t - t^2 = 0$, which gives $t = 0, 2$. So, response falls to the initial level after 2 days.

(c) Rearranging to $R = \frac{1}{2}e^{2t-t^2}$, we differentiate and set the derivative to zero for SPs:

$$\frac{dR}{dt} = (1-t)e^{2t-t^2} = 0.$$

The exponential factor cannot be zero, so $t = 1$ days. Substituting back in, $R = \frac{1}{2}e$, which gives $1.359 \times 10^5 \approx 13600$ white blood cells per millimetre.

(d) To maximise the first derivative, we set the second derivative (using the product rule) to zero:

$$\frac{d^2R}{dt^2} = (2t^2 - 4t + 1)e^{2t-t^2} = 0.$$

Again, the exponential term is always positive, so $2t^2 - 4t + 1 = 0$, which gives $t = 0.293, 1.707$. The rate of change of response is maximised at the first of these, so $t = 0.293$ days (3sf).

2912. (a) The ratio of successive terms is

$$\begin{aligned} \frac{c_{n+1}}{c_n} &= \frac{a_{n+1}b_{n+1}}{a_n b_n} \\ &= \frac{a_{n+1}}{a_n} \times \frac{b_{n+1}}{b_n} \\ &= r_1 r_2. \end{aligned}$$

This is constant, so c_n is a GP.

(b) The common ratios are the same, so we can write $a_n = ar^{n-1}$ and $b_n = br^{n-1}$. The sum is then $a_n + b_n = (a+b)r^{n-1}$, which is geometric with first term $a + b$ and common ratio r.

2913. Starting with the RHS,

$$\begin{aligned} &S(a)\,C(b) + C(a)\,S(b) \\ \equiv\; &\frac{(e^a - e^{-a})(e^b + e^{-b})}{4} + \frac{(e^a + e^{-a})(e^b - e^{-b})}{4} \\ \equiv\; &\frac{2e^a e^b - 2e^{-a}e^{-b}}{4} \\ \equiv\; &\frac{e^{a+b} - e^{-a-b}}{2} \\ \equiv\; &S(a+b), \text{ as required.} \end{aligned}$$

2914. Writing it out longhand, the equation is

$$\frac{1}{x-2} - \frac{1}{x-1} + \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} = 0.$$

Putting this over a common denominator,

$$\frac{x^4 + x^2 + 4}{x^5 - 5x^3 + 4x} = 0.$$

The numerator of the fraction is quadratic in x^2, with discriminant $\Delta = -15 < 0$. So, the equation has no real roots.

2915. Let the decimal part of x be $0.a_1a_2a_3...a_n$, where a_i represents the ith digit and $a_n \neq 0$. We can write this as

$$x = \frac{a_1}{10^1} + \frac{a_2}{10^2} + ... + \frac{a_n}{10^n}.$$

When we square this, the term with the largest negative power is

$$\frac{a_n^2}{10^{2n}}.$$

The numerator cannot end in 0, since $a_n \neq 0$, so the last decimal place is the $2n$th. □

2916. It's correct!

——— Nota Bene ———

It is the *local relative motion of surfaces* which is opposed by friction, not the global absolute motion of an object. Indeed, for most locomotives (trains, cars, bicycles and human beings) it is friction that *causes* motion. All driving forces (not including propulsion forces in aircraft) are in fact frictional forces.

2917. (a) The numerator is zero at $x = 1$.

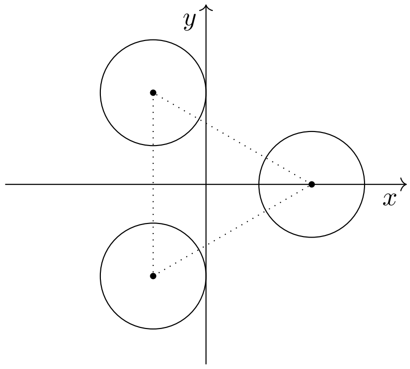
(b) The area of the region is given by

$$A = \int_1^{21} \frac{x-1}{\sqrt{1+3x}}\,dx.$$

Let $u = 1 + 3x$. This gives $dx = \frac{1}{3}du$, and also $x = \frac{1}{3}(u-1)$. The x limits are $x = 1$ and $x = 21$, so the u limits are $u = 4$ and $u = 64$. Enacting the substitution,

$$\begin{aligned} A &= \int_4^{64} \frac{\frac{1}{3}(u-1) - 1}{\sqrt{u}} \, \tfrac{1}{3}du \\ &= \tfrac{1}{9}\int_4^{64} u^{\frac{1}{2}} - 4u^{-\frac{1}{2}}\,du \\ &= \tfrac{1}{9}\left[\tfrac{2}{3}u^{\frac{3}{2}} - 8u^{\frac{1}{2}}\right]_4^{64} \\ &= \tfrac{1}{9}\left(\tfrac{832}{3} + \tfrac{32}{3}\right) \\ &= 32, \text{ as required.} \end{aligned}$$

2918. Angles $\theta = \frac{2\pi i}{3}$ radians $(0, 120, 240°)$ are periodic in $[0, 2\pi)$. Hence, the three centres $(\cos\theta, \sin\theta)$ have rotational symmetry order three around the origin.



So, the centres lie at the vertices of an equilateral triangle, which makes the circles equidistant. □

2919. Rearranging to $x^2 = y + 1$, we substitute for x^2:

$$\begin{aligned} &2x^6 - 7x^4 + 7x^2 - 2 \\ &= 2(y+1)^3 - 7(y+1)^2 + 7(y+1) - 2 \\ &\equiv 2y^3 - y^2 - y. \end{aligned}$$

2920. We condition on the placement of the first bishop, in terms of the distance from the edge of the board. The cases are $\{0, 1, 2, 3\}$ squares from the edge, of which there are $\{28, 20, 12, 4\}$ squares. The number of squares threatened in each case is $\{7, 9, 11, 13\}$. This gives the probability as

$$\frac{28 \cdot 7}{64 \cdot 63} + \frac{20 \cdot 9}{64 \cdot 63} + \frac{12 \cdot 11}{64 \cdot 63} + \frac{4 \cdot 13}{64 \cdot 63} = \frac{5}{36}.$$

2921. Differentiating the equation of a parabola,

$$\begin{aligned} y &= ax^2 + bx + c \\ \Longrightarrow \frac{dy}{dx} &= 2ax + b. \end{aligned}$$

Substituting in, we require

$$2ax + b + ax^2 + bx + c \equiv x + 1.$$

Equating coefficients of x^2 gives $a = 0$. Hence, if there is a solution of the form $y = ax^2 + bx + c$, it must be $y = bx + c$, which is not a parabola.

2922. The LHS may be factorised, giving

$$3^x(3^x - 1)(3^{2x} + 1) > 0.$$

The left-hand and right-hand factors are always positive, so we need only consider $3^x - 1 > 0$, which holds for $x > 0$. So, the solution set is $(0, \infty)$.

2923. Using the change of base formula, we can write

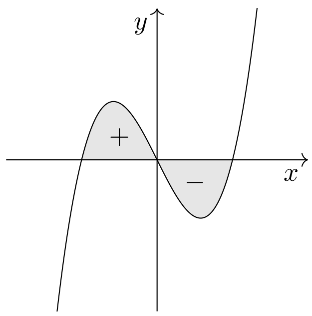
$$y = \log_a x \equiv \frac{\log_b x}{\log_b a} \equiv \log_b x \times \log_a b.$$

Hence, the curve $\log_a x$ is a stretch, scale factor $\log_a b$ in the y direction, of the curve $y = \log_b x$.

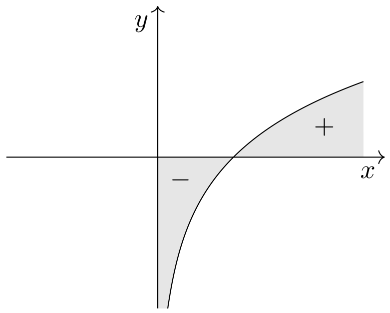
2924. (a) A counterexample is $y = x^{-2}$. The derivative is $\frac{dy}{dx} = -x^{-1}$, which changes sign either side of $x = 0$. But $x = 0$ is not a stationary point, as the curve (and its gradient) are undefined there.

(b) Consider $y = x^4$, at the origin. The second derivative $12x^2$ is zero, but it doesn't change sign: it is positive both sides of $x = 0$. Hence, the stationary point at the origin is an (extra flattened) local minimum.

2925. (a) $\displaystyle\int_{-1}^{1} x^3 - x\,dx = 0$:

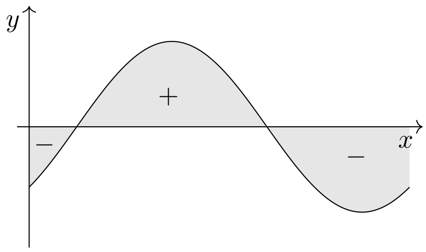


(b) $\displaystyle\int_{0}^{e} \ln x\,dx = 0$:



(c) Putting the integrand in harmonic form,

$$\int_0^{2\pi} \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)\,dx = 0:$$



2926. Equating the squared lengths of the chords,

$$\begin{aligned} & a^2 + a^4 = (1-a)^2 + (1-a^2)^2 \\ \implies & a^2 + a - 1 = 0 \\ \implies & a = \frac{-1 \pm \sqrt{5}}{2}. \end{aligned}$$

Since $0 < a < 1$, we reject the negative root. This gives

$$a = \frac{-1 + \sqrt{5}}{2}.$$

2927. Differentiating implicitly,

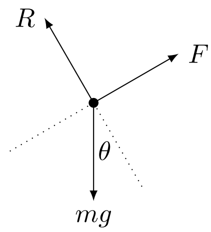
$$\tfrac{1}{2}(x+y)^{-\frac{1}{2}}\left(1 + \tfrac{dy}{dx}\right) - \tfrac{1}{2}(x-y)^{-\frac{1}{2}}\left(1 - \tfrac{dy}{dx}\right) = 0.$$

Multiplying by 2 and setting $\frac{dy}{dx} = 2$,

$$3(x+y)^{-\frac{1}{2}} + (x-y)^{-\frac{1}{2}} = 0.$$

Both terms are strictly positive, so they cannot add to give zero. So, there is no point on the curve with gradient 2, and therefore no line of the form $y = 2x + k$ is tangent to the curve.

2928. The force diagram, for limiting equilibrium, is



Perpendicular to the slope, $R = mg\cos\theta$. And parallel to the plane, $F = mg\sin\theta$. At the angle of friction θ, friction is maximal, at $F_{\max} = \mu R$. This gives

$$\begin{aligned} \mu &= \frac{F}{R} \\ &= \frac{mg\sin\theta}{mg\cos\theta} \\ &\equiv \tan\theta. \end{aligned}$$

Hence, the angle of friction is $\theta = \arctan\mu$.

— Nota Bene —

"Limiting equilibrium" is equilibrium, with the added proviso that any more force in a particular direction would push the object out of equilibrium. In the type of limiting equilibrium in this question, friction is maximal, but the object remains at rest.

2929. There are 9! orders of nine objects. Listing these, we partition the numbers by position:

1,2,3	4,5,6	7,8,9
1,2,3	4,5,6	7,9,8
1,2,3	4,5,6	8,7,9
...	...	...

There are 3! orders within each set of 3. Hence, in the list of 9! orders, we will overcount by a factor of $3! \times 3! \times 3!$. This gives the number of ways as

$$\frac{9!}{3! \times 3! \times 3!} = 1680.$$

2930. Adding the equations, $2s = (u+v)t$. Subtracting the equations, $0 = (u-v)t + at^2$. We rearrange these to

$$\begin{aligned} v + u &= \frac{2s}{t}, \\ v - u &= at. \end{aligned}$$

Multiplying the equations,

$$\begin{aligned} &(v+u)(v-u) = 2as \\ \implies &v^2 = u^2 + 2as, \text{ as required.} \end{aligned}$$

2931. Writing the series out longhand, we have

$$(a-b)\big(a^{n-1} + a^{n-2}b + ... + ab^{n-2} + b^{n-1}\big).$$

Multiplying out the brackets, this is

$$\begin{aligned} &a^n + a^{n-1}b + ... + a^2b^{n-2} + ab^{n-1} \\ &\quad -\big(a^{n-1}b + a^{n-2}b^2 + ... + ab^{n-1} + b^n\big). \end{aligned}$$

All but two terms cancel, leaving $a^n - b^n$.

— Nota Bene —

This result is the generalisation of the difference of two squares: a difference of two nth powers $a^n - b^n$ always has a factor of $a - b$.

2932. (a) By the chain rule, the first derivative is

$$\frac{dy}{dx} = \frac{\cos(\ln x)}{x}.$$

Evaluating at $x = 1$, this gives $m = 1$. The equation of the tangent is $y = x - 1$.

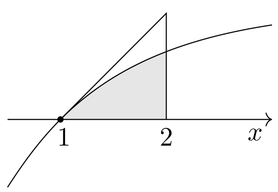
(b) The second derivative is

$$\frac{d^2y}{dx^2} = -\frac{\sin(\ln x) + \cos(\ln x)}{x^2}.$$

On $(1, 2]$, the range of $\ln x$ is $(0, \ln 2]$. Since $\ln 2 < \frac{\pi}{2}$, both $\sin(\ln x)$ and $\cos(\ln x)$ are +ve over the given domain; x^2 is also +ve, so the second derivative is negative. Therefore, the curve is concave.

(c) The curve is concave on $[1, 2]$, so, since $\frac{dy}{dx} = 1$ at $x = 1$, we know that $\frac{dy}{dx} < 1$ for $x \in (1, 2]$. This puts the curve below T.

(d) The region under T, over the domain $[1, 2]$, is a triangle, with base 1 and height 1.



Its area is $\frac{1}{2}$. Since the curve lies under T, we know that $I < \frac{1}{2}$.

2933. Starting with the RHS,

$$\begin{aligned} &\frac{1 + \tan^2\theta}{1 - \tan^2\theta} \\ \equiv\ &\frac{1 + \frac{\sin^2\theta}{\cos^2\theta}}{1 - \frac{\sin^2\theta}{\cos^2\theta}}. \end{aligned}$$

Multiplying top and bottom by $\cos^2\theta$, this is

$$\begin{aligned} &\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta - \sin^2\theta} \\ \equiv\ &\frac{1}{\cos 2\theta} \\ \equiv\ &\sec 2\theta, \text{ as required.} \end{aligned}$$

2934. The first counter can be placed wlog. Then, the probability that the second is in a different row and column is $\frac{4}{8}$. Then the probability that the third is also in a different row and column is $\frac{1}{7}$. So, the probability is $p = \frac{4}{8} \times \frac{1}{7} = \frac{1}{14}$.

— Alternative Method —

The possibility space consists of ${}^9\text{C}_3$ equally likely ways of placing the counters. In each successful outcome, there is one counter in each row. There are three locations in the first row, then two in the second row and one in the third. This gives

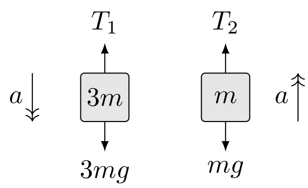
$$p = \frac{3 \times 2 \times 1}{{}^9\text{C}_3} = \tfrac{1}{14}.$$

2935. The error is not with the product rule, but rather with (a failure to use) the chain rule.

The factor $\cos(3x + 1)$ has not been differentiated correctly. By the chain rule, its derivative should be $-3\sin(3x + 1)$, the derivative of $(3x + 1)$ being 3. A corrected version is:

$$\begin{aligned} &y = e^x \cos(3x+1) \\ \implies &\frac{dy}{dx} = e^x\cos(3x+1) - 3e^x \sin(3x+1). \end{aligned}$$

2936. The force diagrams are as follows:



(a) Using NIII, the total downwards force on the pulley is $R = T_1 + T_2$, so the tensions differ by $\frac{1}{4}R = (T_1 + T_2)$. Clearly T_1 must be larger, so

$$\begin{aligned} &T_1 - T_2 = \tfrac{1}{4}(T_1 + T_2) \\ \implies &\tfrac{3}{4}T_1 = \tfrac{5}{4}T_2 \\ \implies &3T_1 = 5T_2, \text{ as required.} \end{aligned}$$

(b) The vertical equations of motion are

$$\text{①}\ 3mg - T_1 = 3ma,$$
$$\text{②}\ T_2 - mg = ma.$$

Taking $3 \times \text{①} + 5 \times \text{②}$,

$$\begin{aligned} & 9mg - 3T_1 + 5T_2 - 5mg = 9ma + 5ma \\ \Longrightarrow\ & 4mg = 14ma \\ \Longrightarrow\ & a = \tfrac{2}{7}g, \text{ as required.} \end{aligned}$$

2937. Assume, for a contradiction, that c is even and a is odd, where $a^2 + b^2 = c^2$. Then b must also be odd. Expressing this algebraically, for $p, q, r \in \mathbb{N}$,

$$\begin{aligned} & a^2 + b^2 = c^2 \\ \Longrightarrow\ & (2p+1)^2 + (2q+1)^2 = (2r)^2 \\ \Longrightarrow\ & 4(p^2 + p + q^2 + q) + 2 = 4r^2. \end{aligned}$$

The LHS is two greater than a multiple of 4, while the RHS is a multiple of 4. This is a contradiction. So, if c is even, then so are a and b. □

2938. Each RHS has a factor of x. Hence, $(0, 0)$ satisfies the equations. If $x \neq 0$, then we can make $\frac{y}{x}$ the subject of each equation. Equating these,

$$\begin{aligned} & e^x = 6e^{-x} - 1 \\ \Longrightarrow\ & e^{2x} + e^x - 6 = 0 \\ \Longrightarrow\ & (e^x + 3)(e^x - 2) = 0 \\ \Longrightarrow\ & e^x = -3, 2. \end{aligned}$$

Since $e^x > 0$, this gives $x = \ln 2$. So, the solution points are $(0, 0)$ and $(\ln 2, 2\ln 2)$.

2939. (a) By the quotient rule, the derivative is

$$\frac{dy}{dx} = \frac{-10}{(x-2)^2}.$$

Evaluating at $x = 0$, we get $m = -\frac{5}{2}$. So, the equation of the tangent is

$$\begin{aligned} & y - 1 = -\tfrac{5}{2}(x - 0) \\ \Longrightarrow\ & y = 1 - \tfrac{5}{2}x. \end{aligned}$$

(b) The roots of the equation $\mathrm{f}(x) = \mathrm{f}_1(x)$ are the x coordinates of the intersections of $y = \mathrm{f}(x)$ and $y = \mathrm{f}_1(x)$. Since $y = \mathrm{f}'(x)$ is tangent to $y = \mathrm{f}(x)$, there is a point of intersection which is a point of tangency. At a point of tangency, there is a repeated root.

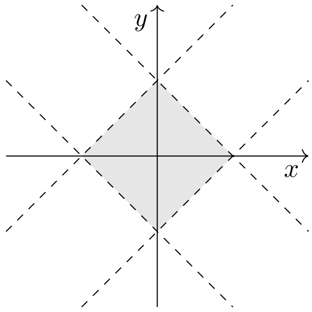
2940. The boundary equations of these inequalities are

$$(x+y)^2 = a,$$
$$(x-y)^2 = a.$$

Taking square roots,

$$x + y = \pm\sqrt{a},$$
$$x - y = \pm\sqrt{a}.$$

These are two pairs of parallel lines.



The axis intercepts are at $\pm\sqrt{a}$. So, the area of the square is $4 \times \frac{1}{2}\left(\sqrt{a}\right)^2 = 2a$.

2941. We assume that rotations and reflections of the bracelet are indistinguishable. Then, by number of white beads, the first four configurations are

(0) : BBBBB
(1) : WBBBB
(2) : WWBBB, WBWBB.

There are then four symmetrical configurations with white and black interchanged. Hence, there are eight configurations overall, as required.

2942. Differentiating implicitly,

$$\begin{aligned} & x^2 - \sqrt{x+y} = 1 \\ \Longrightarrow\ & 2x - \tfrac{1}{2}(x+y)^{-\frac{1}{2}}\left(1 + \frac{dy}{dx}\right) = 0 \\ \Longrightarrow\ & 4x(x+y)^{\frac{1}{2}} - \left(1 + \frac{dy}{dx}\right) = 0 \\ \Longrightarrow\ & \frac{dy}{dx} = 4x(x+y)^{\frac{1}{2}} - 1. \end{aligned}$$

So, $\dfrac{dy}{dx} = 4x\sqrt{x+y} - 1$, as required.

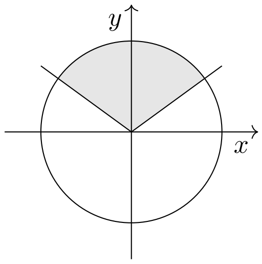
2943. (a) Together, the four curved sections make up one full circle. So, the total length is

$$\begin{aligned} l &= 4 \times 2r + 2\pi r \\ &\equiv 2r(4 + \pi). \end{aligned}$$

(b) By NIII, the two instances of any frictional force between the upper two logs must act one vertically upwards and the other vertically downwards. But the situation is symmetrical, so this asymmetry is impossible. Hence, the frictional forces must be zero.

(c) Since there is no vertical friction acting on the upper logs, the only vertical forces are weight and tension. Vertical equilibrium for one of the upper logs is $R - W - 2W = 0$, giving $R = 3W$.

2944. The possibility space, with the successful region shaded, is:



The angle of inclination of the mod graph, in the positive quadrant, is

$$\arctan\sqrt{5-2\sqrt{5}} = \tfrac{\pi}{5}.$$

So, the angle subtended at the origin by the shaded area is $\frac{3\pi}{5}$. For the probability, we divide the area of the successful region by the area of the circle, which gives $p = \frac{3\pi}{5} \div 2\pi = \frac{3}{10}$.

2945. Assume, for a contradiction, that four distinct points on the cubic graph $y = \text{f}(x)$ lie on the parabola $y = \text{g}(x)$. Then the equation $\text{f}(x) = \text{g}(x)$ has four distinct roots. But this equation is cubic, so has at most three roots. This is a contradiction. Hence, no four distinct points on a cubic $y = \text{f}(x)$ lie on the same parabola $y = \text{g}(x)$. QED.

2946. Integrating by inspection,

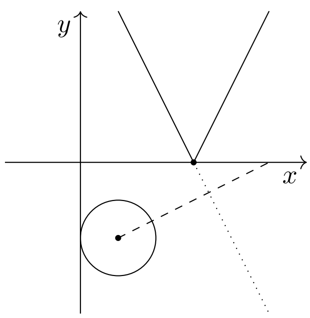
$$\int \text{f}'(x)e^{\text{f}(x)}\,dx = e^{\text{f}(x)} + c.$$

The justification is differentiation of $e^{\text{f}(x)} + c$ by the chain rule.

2947. Completing the square, the circle is

$$(x-1)^2 + (y+2)^2 = 1.$$

So, it has centre $(1, -2)$ and radius 1. The mod graph has a vertex at $(3, 0)$, and gradient ∓ 2. Sketching these,



The perpendicular to the LH branch which passes through the centre (dashed above) has equation $y + 2 = \frac{1}{2}(x-1)$. At $x = 3$, this is at $y = -1$, which is below the vertex of the mod graph. Hence, the vertex is the closest point to the circle.

By Pythagoras, the distance from the centre to the vertex is $2\sqrt{2}$, so the shortest distance between the graphs is $2\sqrt{2} - 1$.

2948. (a) The limiting total displacement is given by the area under the above graph from $t = 0$ to $t = T$, in the limit that $T \to \infty$. This is

$$S = \int_0^\infty \frac{1}{e^t + 1}\,dt$$

Let $u = e^t + 1$, so $du = e^t\,dt$. Replacing e^t in this,

$$dt = \frac{1}{u-1}\,du.$$

The u limits are $u = 2$ to $u = \infty$. Enacting the substitution,

$$S = \int_2^\infty \frac{1}{u(u-1)}\,du.$$

(b) For partial fractions,

$$\frac{1}{u(u-1)} \equiv \frac{A}{u} + \frac{B}{u-1}$$
$$\Longrightarrow 1 \equiv A(u-1) + Bu.$$

Equating constant terms, $A = -1$; equating coefficients of u, $B = 1$. So, the integral is

$$\begin{aligned} S &= \int_2^\infty \frac{1}{u-1} - \frac{1}{u}\,du \\ &= \Big[\ln|u-1| - \ln|u|\Big]_2^\infty \\ &= \Big[\ln\left|\tfrac{u-1}{u}\right|\Big]_2^\infty. \end{aligned}$$

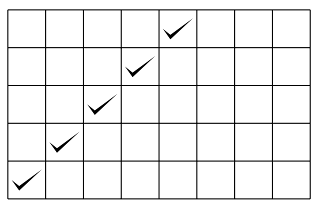
At the upper, infinite limit, $\frac{u-1}{u}$ tends to 1. Since $\ln 1 = 0$,

$$\begin{aligned} S &= (0) - \left(\ln\tfrac{1}{2}\right), \\ &= \ln 2, \text{ as required.} \end{aligned}$$

2949. If we roll the m-sided die first, then its value can be observed without loss of generality. Having noted this value, the probability that the n-sided die shows the same score is $\frac{1}{n}$.

— Alternative Method —

The possibility space is an $m \times n$ grid of equally likely outcomes. The successful outcomes lie on the leading diagonal. Since $m < n$, this consists of m outcomes:



So, the probability is $\frac{m}{mn}$, which is $\frac{1}{n}$.

2950. This is false. The function $\text{f}(x) = 1/x$ has a sign change at $x = 0$, but $1/x$ is undefined at $x = 0$. So, $a = -1$, $b = 1$ and $\text{f}(x) = 1/x$ is a counterexample which disproves the student's claim.

——— Nota Bene ———

Although it doesn't hold in general, the student's statement does hold for functions which have no discontinuities, such as polynomials.

2951. Using the first Pythagorean trig identity,

$$\begin{aligned} &\sin^3 x + \cos^2 x + 1 = 0 \\ \Longrightarrow\ &\sin^3 x - \sin^2 x + 2 = 0. \end{aligned}$$

This is a cubic in $\sin x$. Using a polynomial solver, $\sin x = -1$. Hence, $x = \frac{3\pi}{2}$.

2952. Assume, for a contradiction, that $\log_3 5 = p/q$, for $p, q \in \mathbb{N}$. Rewriting this as an index statement,

$$3^{\frac{p}{q}} = 5.$$

Raising both sides to the power q,

$$3^p = 5^q.$$

The LHS has no prime factors of 5, while the RHS has no factors of 3. Hence, $p = q = 0$. This is a contradiction. Therefore, $\log_3 5$ is irrational. □

2953. (a) Splitting the integral up,

$$\int_{-1}^{1} \text{f}(x)\,dx = \int_{-1}^{0} \text{f}(x)\,dx + \int_{0}^{1} \text{f}(x)\,dx.$$

Since the curve has rotational symmetry, the values of these two integrals are $-k$ and k. So, the value of the original integral is zero.

(b) Rotational symmetry gives $\text{f}(-x) \equiv -\text{f}(x)$, so that $\text{f}(-2x) \equiv -\text{f}(2x)$. Hence, the integrand is $2\,\text{f}(2x)$. Let $u = 2x$, which gives $du = 2\,dx$. Enacting the substitution, the integral is now

$$\int_0^1 \text{f}(u)\,du.$$

This has value k.

2954. By the chain rule,

$$\frac{dx_n}{dx_1} \equiv \frac{dx_n}{dx_{n-1}} \cdot \frac{dx_{n-1}}{dx_{n-2}} \cdot \ldots \cdot \frac{dx_2}{dx_1}.$$

Each of the derivatives on the RHS has value 2. So,

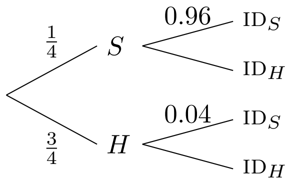
$$\frac{dx_n}{dx_1} = 2^{n-1}.$$

Integrating with respect to x_1,

$$x_n = 2^{n-1}x_1 + c, \text{ as required.}$$

2955. (a) The prosecutor's argument relies on equal numbers of each kind of car. If there are, for example, many more hatchbacks than saloons, then a car identified as a saloon is more likely to be a hatchback wrongly identified than a saloon correctly identified.

(b) Conditioning on the type of car, the possibility space is



Restricting the possibility space to only cars identified as saloons,

$$\mathbb{P}(H \mid \text{ID}_S) = \frac{\frac{3}{4} \times 0.04}{\frac{1}{4} \times 0.96 + \frac{3}{4} \times 0.04} = \frac{1}{9}.$$

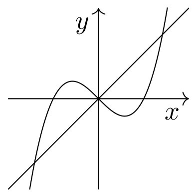
The true probability is 1 in 9, not 1 in 25.

2956. The denominator of the fraction is a quadratic in $\ln x$. Completing the square,

$$1 - \ln x + (\ln x)^2 \equiv \left(\ln x - \tfrac{1}{2}\right)^2 + \tfrac{3}{4}.$$

Since $\ln x$ can take the value $\frac{1}{2}$, the minimum value of this expression is $\frac{3}{4}$. Hence, the maximum value of the original fraction is $\frac{4}{3}$.

2957. The $+c$ translates the entire problem by vector $c\mathbf{j}$. This doesn't affect the areas. So, we can set $c = 0$ and proceed without loss of generality. Consider $y = x^3 - x$.



This curve's point of inflection is at the origin. The derivative is $\frac{dy}{dx} = 3x^2 - 1$, so the tangent at the origin is $y = -x$ and the normal is $y = x$. Solving for intersections,

$$\begin{aligned} &x^3 - x = x \\ \Longrightarrow\ &x(x^2 - 2) = 0 \\ \Longrightarrow\ &x = 0, \pm\sqrt{2}. \end{aligned}$$

Cubics are rotationally symmetrical around their point of inflection, so the areas enclosed must be equal. In square units, each region has area

$$\begin{aligned} A &= \int_0^{\sqrt{2}} x - (x^3 - x)\,dx \\ &= \left[x^2 - \tfrac{1}{4}x^4\right]_0^{\sqrt{2}} \\ &= (2 - 1) - (0 - 0) \\ &= 1, \text{ as required.} \end{aligned}$$

2958. In completed-squared form, the graph is

$$y = -\left(x - \frac{a+b}{2}\right)^2 + \frac{(a-b)^2}{4}.$$

Equating to zero for roots,

$$\begin{aligned} -\left(x - \frac{a+b}{2}\right)^2 + \frac{(a-b)^2}{4} &= 0 \\ \implies x - \frac{a+b}{2} &= \pm\frac{a-b}{2} \\ \implies x &= \frac{a+b}{2} \pm \frac{a-b}{2} \\ &\equiv a, b, \text{ as required.} \end{aligned}$$

2959. (a) Lines ② and ③ have gradients $m_2 = 2/5$ and $m_3 = -5/2$. So, we have a right-angled triangle with line ① as the hypotenuse.

(b) Vertex A is the intersection of the first two lines. Solving simultaneously, it is $(0, -9/5)$.

(c) The sides through A are at $\arctan 2/5$ above and $\arctan 1$ below the positive x direction. Hence, the total angle is $45° + \arctan 2/5$.

(d) The ratio of the two perpendicular sides is given by $\tan(45° + \arctan 2/5) = 7/3$.

(e) Let $d > 0$ be the length of the shortest side. The area is

$$\tfrac{1}{2}d \times \tfrac{7}{3}d \equiv \tfrac{7}{6}d^2.$$

Setting this to $\frac{406}{75}$, we solve to get $d = \frac{2\sqrt{29}}{5}$. Resolving this distance into components,

$$\begin{aligned} x &= d\cos(\arctan 2/5) = 2, \\ y &= d\sin(\arctan 2/5) = 4/5. \end{aligned}$$

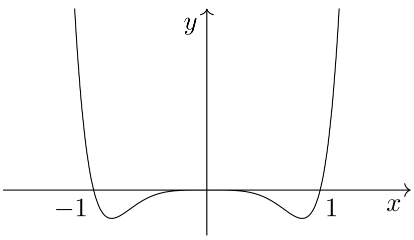
Hence, triangle T has a vertex at $(2, -1)$. This gives $k = 8$.

2960. There are 7! arrangements of $\text{PIZ}_1\text{Z}_2\text{AZ}_3\text{Z}_4$. In the list of 7! arrangements, every word (e.g. PIZZAZZ) will appear 4! times, once for each arrangement of $\text{Z}_1\text{Z}_2\text{Z}_3\text{Z}_4$. Hence, we overcount by a factor of 4!. This gives $\frac{7!}{4!} = 210$.

2961. In factorised form, the graph is

$$y = x^4(x^2+1)(x-1)(x+1).$$

This gives a single root at $x = -1$, a quadruple root (turning point) at $x = 0$ and a single root at $x = 1$. The quadratic factor has no real roots. The overall shape is a positive polynomial of even degree:



2962. The given facts are

$$\begin{aligned} ar &= a + d, \\ ar^2 &= a + 4d. \end{aligned}$$

Eliminating d, we know that

$$\begin{aligned} ar^2 - 4ar + 3a &= 0 \\ \implies a(r^2 - 4r + 3) &= 0. \end{aligned}$$

We can rule out $a = 0$: it would give $d = 0$, and we are told that the AP is non-constant. So,

$$\begin{aligned} r^2 - 4r + 3 &= 0 \\ \implies r &= 1, 3. \end{aligned}$$

The GP is also non-constant, so we reject $r = 1$. This leaves $r = 3$.

2963. The curve $y = x^2\sqrt{x+2}$ has domain of definition $[-2, \infty)$, a single root at $x = -2$ and a double root at $x = 0$. These are consistent with the graph. For SPs,

$$\begin{aligned} 2x(x+2)^{\frac{1}{2}} + \tfrac{1}{2}x^2(x+2)^{-\frac{1}{2}} &= 0 \\ \implies x\big(2(x+2) + \tfrac{1}{2}x\big) &= 0 \\ \implies x &= 0, -\tfrac{8}{5}. \end{aligned}$$

The fact that $y \geq 0$ everywhere guarantees that the SP at $x = -8/5$ is a maximum, and that the SP at the origin is a minimum. The behaviour $x \to \infty, y \to \infty$ is also correct. So, the equation is consistent with the graph shown.

2964. Consider the E and the A as a single item. There are 4! ways of rearranging this item with the other three letters. For each of these ways, we can write EA or AE. This gives $24 \times 2 = 48$ arrangements, as required.

2965. This is a quadratic in $\sqrt{x}$:

$$\begin{aligned} 3\sqrt{x} + \frac{2}{\sqrt{x}} &= 7 \\ \implies 3x - 7\sqrt{x} + 2 &= 0 \\ \implies (3\sqrt{x} - 1)(\sqrt{x} - 2) &= 0 \\ \implies x &= \tfrac{1}{9}, 4. \end{aligned}$$

2966. The mean of $X \sim B(6, 0.27)$ is $6 \times 0.27 = 1.62$. The integer closest to the mean is 2, but the mode is not 2, as shown below:

$$\begin{aligned} \mathbb{P}(X = 1) &\approx 0.34, \\ \mathbb{P}(X = 2) &\approx 0.31. \end{aligned}$$

Hence, $X \sim B(6, 0.27)$ disproves the claim.

2967. The constant term 33 has factors 3 and 11. So, we look for a factorisation of the form

$$\begin{aligned}&x^4+2x^3+15x^2+14x+33\\ \equiv\ &(x^2+ax+3)(x^2+bx+11).\end{aligned}$$

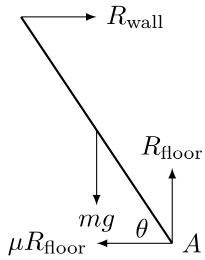
Equating coefficients of x^3 and x,

$$\begin{aligned}a+b&=2,\\ 3b+11a&=14.\end{aligned}$$

Solving these, we get $a=b=1$. The factorisation can be verified with the x^2 term.

Looking for roots, the discriminants are $\Delta_1=-11$ and $\Delta_2=-120$. These are both negative, so the quartic has no real roots.

2968. (a) The force diagram is



Taking the ladder to have length $2l$,

$$\begin{aligned}\updownarrow &: R_{\text{floor}}-mg=0\\ \leftrightarrow &: R_{\text{wall}}-\mu R_{\text{floor}}=0\\ \overset{\frown}{A} &: R_{\text{wall}}\cdot 2l\sin\theta-mg\cdot l\cos\theta=0.\end{aligned}$$

Solving, we get $R_{\text{floor}}=mg$, so $R_{\text{wall}}=\mu mg$. Dividing by lmg, the moments equation is

$$\begin{aligned}2\mu\sin\theta&=\cos\theta\\ \Longrightarrow \mu&=\tfrac{1}{2}\cot\theta.\end{aligned}$$

This value is limiting friction, so is a lower bound on μ. Hence, $\mu\geq\frac{1}{2}\cot\theta$, as required.

(b) As $\theta\to 0$, the value of $\cot\theta$ and hence the minimum value of μ grows asymptotically. In other words, if the ladder is nearly horizontal, a very large coefficient of friction is needed to keep it stable.

2969. By the reverse chain rule,

$$\int\cos x\sin^2 x\,dx=\tfrac{1}{3}\sin^3 x+c.$$

Nota Bene

Integrations by inspection are best understood in reverse, by differentiating the answer. For further elucidation (the long way round), you could retry this integral with the substitution $u=\sin x$.

2970. Writing everything in terms of 2^x,

$$\begin{aligned}&2^{x+1}+4^{x+1}=8^x+15\\ \Longrightarrow\ &\left(2^x\right)^3-4\cdot\left(2^x\right)^2-2\cdot\left(2^x\right)+15=0.\end{aligned}$$

This is a cubic in 2^x. The factor theorem tells us that, since $x=\log_2 3$ is a root, (2^x-3) must be a factor. Taking it out,

$$\left(2^x-3\right)\left(\left(2^x\right)^2-2^x-5\right)=0.$$

Looking for roots of the quadratic factor,

$$\begin{aligned}2^x&=\tfrac{1\pm\sqrt{21}}{2}\\ \Longrightarrow x&=\log_2\tfrac{1\pm\sqrt{21}}{2}\\ &=\log_2\left(1\pm\sqrt{21}\right)-1.\end{aligned}$$

Since $\log_2$ cannot take negative inputs, only the positive of these produces a root. So, the solution is

$$x=\log_2 3,\log_2\left(1+\sqrt{21}\right)-1.$$

2971. (a) The standard trapezium rule gives

$$A_1=\tfrac{1}{2}\cdot\tfrac{1}{2}\left(0+2\cdot\tfrac{1}{4}+1\right)=0.375.$$

(b) Let the central point be (p,p^2). Equating the squared lengths of the chords from $(0,0)$ and $(1,1)$,

$$\begin{aligned}&p^2+p^4=\left(1-p\right)^2+\left(1-p^2\right)^2\\ \Longrightarrow\ &p^2+p-1=0\\ \therefore\ &p=\tfrac{-1+\sqrt{5}}{2}.\end{aligned}$$

Using this value, the approximation is

$$A_2=\tfrac{1}{2}p\left(p^2\right)+\tfrac{1}{2}(1-p)\left(p^2+1\right)\approx 0.382.$$

The true value is $0.\dot{3}$, so the approximation $A_2\approx 0.382$ is further from the true value than $A_1=0.375$. The standard trapezium rule gives the better approximation.

2972. Using the parametric differentiation formula,

$$\frac{dy}{dx}\equiv\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{-4\sin 2t}{\cos t}.$$

The gradient of the vector $\mathbf{i}+4\mathbf{j}$ is 4:

$$\begin{aligned}&\frac{-4\sin 2t}{\cos t}=4\\ \Longrightarrow\ &-\sin 2t=\cos t\\ \Longrightarrow\ &2\sin t\cos t+\cos t=0\\ \Longrightarrow\ &\cos t(2\sin t+1)=0\\ \Longrightarrow\ &\cos t=0\text{ or }\sin t=-\tfrac{1}{2}.\end{aligned}$$

The first root $t>0$ is at $t=\frac{\pi}{2}$.

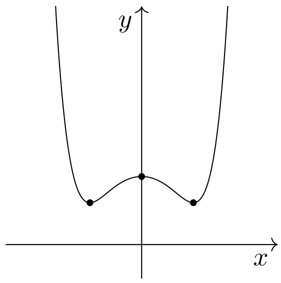
2973. The curve is a positive sextic. Setting the first derivative to zero for SPs,

$$\begin{aligned} 6x^5 - 2x &= 0 \\ \Longrightarrow x(3x^4 - 1) &= 0 \\ \Longrightarrow x &= 0, \pm\sqrt[4]{1/3}. \end{aligned}$$

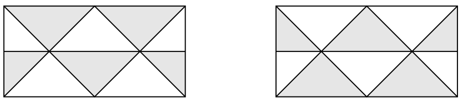
Testing these values, the SPs are at

x	0	$\sqrt[4]{1/3}$	$-\sqrt[4]{1/3}$
y	1	0.6151	0.6151

Since all of the SPs have positive y values, and the curve is a positive polynomial of even degree, it doesn't cross the x axis:

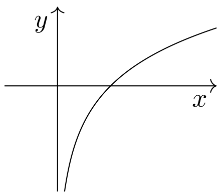


2974. Five regions are shaded, so the possibility space is a set of ${}^{10}\text{C}_5 = 252$ equally likely outcomes. For success, every other region is shaded, so there are only two successful outcomes:



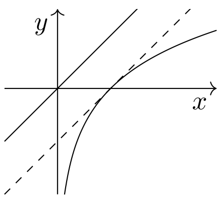
Hence, the probability is $\frac{2}{252} = \frac{1}{126}$.

2975. (a) The graph of $y = \ln x$ is



(b) The derivative is $y = \frac{1}{x}$. Setting this to 1 gives $x = 1$. At this point, the tangent is $y = x - 1$.

(c) Since $y = \ln x$ is concave everywhere (curving downwards), any line of the form $y = x + k$, where $k > -1$, will not intersect $y = \ln x$.



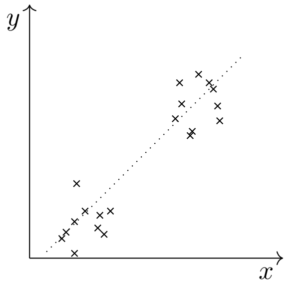
This includes the line $y = x$, as shown above. So, since $\ln x = x$ has no roots, the iteration $x_{n+1} = \ln x_n$ has no fixed points.

2976. The mean of the AP must be the interior angle of a regular hexagon, which is $2\pi/3$ radians. The lower bound (which is not attainable) on the smallest angle is 0, which is 2.5 common differences away from the mean. This puts the upper bound on the common difference as

$$d = \frac{2\pi}{3} \div 2.5 = \frac{4\pi}{15}.$$

Hence, the set of possible values for the third largest angle is $[2\pi/3, 2\pi/3 + d/2)$. Substituting for d, this simplifies to $[2\pi/3, 4\pi/5)$.

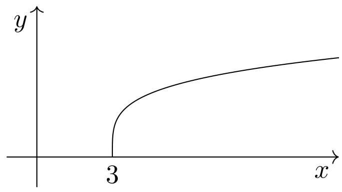
2977. If each sample is a subpopulation with small spread and no correlation, yet the sample means $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$ lie far away from each other along a line of positive gradient, then the combined sample will have positive correlation, even though the individual samples do not:



2978. This is a geometric series with first term $a = 1$, common ratio $r = 3$ and $k + 1$ terms. Using the standard formula, the sum is

$$\begin{aligned} S &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{3^{k+1} - 1}{2}. \end{aligned}$$

2979. The fourth-root graph is akin to the square-root graph. We then translate by vector $3\mathbf{i}$:



2980. (a) Vertically, $-1 = (3\sin\theta)t - 5t^2$. Horizontally, $2 = (3\cos\theta)t$. Substituting the latter into the former,

$$\begin{aligned} -1 &= 3\sin\theta\left(\frac{2}{3\cos\theta}\right) - 5\left(\frac{2}{3\cos\theta}\right)^2 \\ \Longrightarrow -1 &= 2\tan\theta - \frac{20}{9\cos^2\theta} \\ \Longrightarrow 20\sec^2\theta &- 18\tan\theta - 9 = 0. \end{aligned}$$

(b) Substituting the identity $\sec^2\theta \equiv \tan^2\theta + 1$ gives a quadratic in $\tan\theta$:

$$\begin{aligned} &20(\tan^2\theta+1) - 18\tan\theta - 9 = 0 \\ \Longrightarrow\ &20\tan^2\theta - 18\tan\theta + 11 = 0. \end{aligned}$$

This has discriminant

$$\Delta = 18^2 - 4\cdot 20\cdot 11 = -556 < 0.$$

The quadratic has no real roots, so it is not possible to reach the target.

2981. There are $2^6 = 64$ ways of colouring the six edges. Of these, there are six ways with RRRRBB (it can be cycled to start at any one of six edges) and six with BBBBRR. Hence, $p = \frac{12}{64} = \frac{3}{16}$.

2982. When $|x-1|$ is small, then x is close to 1. At the point $(1,0)$, the curve $y = \ln x$ is approximated by its tangent at that point. The derivative is $\frac{dy}{dx} = \frac{1}{x}$, which gives $m = 1$, so the equation of the tangent is $y = x - 1$. Hence, when $|x-1|$ is small, $\ln x \approx x - 1$. □

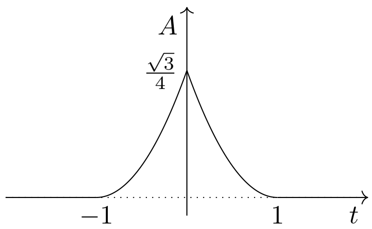
2983. We integrate the jerk three times. The result is polynomial, so each constant of integration is the initial value of the variable in question. Using Newton's dot notation for the time derivative:

$$\begin{aligned} &\dddot{x} = j \\ \Longrightarrow\ &\ddot{x} = a_0 + jt \\ \Longrightarrow\ &\dot{x} = v_0 + a_0 t + \tfrac{1}{2}jt^2 \\ \Longrightarrow\ &x = x_0 + v_0 t + \tfrac{1}{2}a_0 t^2 + \tfrac{1}{6}jt^3. \end{aligned}$$

2984. For $t \notin (-1,1)$, the area is 0. Within $(-1,1)$, the shaded region is an equilateral triangle of linearly increasing/decreasing side length. The area of an equilateral triangle is

$$A = \tfrac{\sqrt{3}}{4}l^2,$$

which is maximised, in our example, at $l = 1$. This occurs at $t = 0$, when the triangles overlap fully. So, the graph of A against t is as follows:



2985. (a) $\ln\frac{1}{x} \equiv \ln x^{-1} \equiv -\ln x$.

(b) $\log_{e^2} x \equiv \log_{\sqrt{e^2}} \sqrt{x} \equiv \frac{1}{2}\ln x$.

(c) $\log_{\frac{1}{e}} x \equiv \log_{\frac{1}{e}^{-1}} \equiv x^{-1}\ln x^{-1} \equiv -\ln x$.

2986. The endpoints of the chord have coordinates $A : (1,0)$ and $B : (-1,2)$. The equation of chord AB is $y = -x + 1$. The equation for intersections is

$$\begin{aligned} &x^2 - x^3 = -x + 1 \\ \Longrightarrow\ &x^3 - x^2 - x + 1 = 0 \\ \Longrightarrow\ &(x-1)^2(x+1) = 0. \end{aligned}$$

Since $(x-1)$ is a squared factor, $x = 1$ is a double root, which signifies a point of tangency.

2987. Each equation is a quadratic in t:

$$\begin{aligned} &t^2 - 2t + x = 0, \\ &t^2 + 2t - y = 0. \end{aligned}$$

Equating the two expressions for t given by the quadratic formula, we get the following, in which the $\pm$ signs are independent of each other:

$$\begin{aligned} &\frac{2 \pm \sqrt{4-4x}}{2} = \frac{-2 \pm \sqrt{4+4y}}{2} \\ \Longrightarrow\ &2 \pm \sqrt{1-x} = \pm\sqrt{1+y}. \end{aligned}$$

— Nota Bene —

Not all of the combinations of $\pm$ signs produce points. $2 + \sqrt{1-x} = -\sqrt{1+y}$ doesn't, as the LHS is positive and the RHS isn't. But this doesn't mean the Cartesian equation is wrong. Between them, the other three options cover all points on the original curve (which is a parabola). Sketching with a graphing calculator is instructive.

2988. Starting with the RHS,

$$\begin{aligned} &T_a T_b + T_{a-1}T_{b-1} \\ \equiv\ &\tfrac{1}{2}a(a+1)\cdot\tfrac{1}{2}b(b+1) + \tfrac{1}{2}(a-1)a\cdot\tfrac{1}{2}(b-1)b \\ \equiv\ &\tfrac{1}{4}ab\big((a+1)(b+1) + (a-1)(b-1)\big) \\ \equiv\ &\tfrac{1}{4}ab(2ab+2) \\ \equiv\ &\tfrac{1}{2}ab(ab+1) \\ \equiv\ &T_{ab}, \text{ as required.} \end{aligned}$$

2989. (a) The lengths of the rectangle are p and $1 - p^2$. Hence, the area is given by $A = p - p^3$. This is a negative cubic, so, as $p \to \infty$, $A \to -\infty$ and is therefore unbounded.

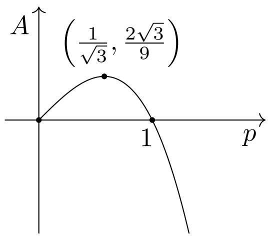
(b) Setting $A = 0$, we solve to find $p = 0, \pm 1$. But $p \in [0, \infty)$, so the area is zero for two values, $p = 0, 1$.

(c) To optimise the area, we set its derivative to zero:

$$\begin{aligned} &\frac{dA}{dp} = 1 - 3p^2 = 0 \\ \therefore\quad &p = \tfrac{1}{\sqrt{3}}. \end{aligned}$$

Substituting in, $A_{\max} = \frac{2\sqrt{3}}{9}$.

(d) For $p \in [0, \infty)$, the graph of A against p is a negative cubic with roots at $p = 0, 1$, and a local maximum in between:



2990. Differentiating the function,

$$\begin{aligned} &\mathrm{f}(x) = x^2 - e^{x^2} + 2 \\ \Longrightarrow\ &\mathrm{f}'(x) = 2x - 2xe^{x^2}. \end{aligned}$$

This gives $\mathrm{f}'(0) = 0$, so the graph $y = x^2 - e^{x^2} + 2$ has a stationary point at $x = 0$. The N-R method looks for an intersection of the tangent with the x axis. But if the gradient is zero then there is no such intersection. Hence, the method breaks down when starting at such a stationary point.

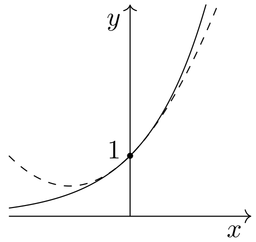
2991. (a) The exponential function $x \mapsto e^x$ differentiates to itself. And $e^0 = 1$. Hence, every derivative of the exponential function is 1 at $x = 0$.

(b) Comparing the derivatives of g at zero,

$$\begin{aligned} \mathrm{g}(0) &= a_0 = 1, \\ \mathrm{g}'(0) &= a_1 = 1, \\ \mathrm{g}''(0) &= \tfrac{1}{2}a_2 = 1. \end{aligned}$$

So, $a_0 = 1$, $a_1 = 1$ and $a_2 = \frac{1}{2}$.

(c) The exponential graph $y = e^x$ (shown solid) and its approximation $y = 1 + x + \frac{1}{2}x^2$ are:



2992. Call the perpendicular lengths a, b:

$$\begin{aligned} &\tfrac{1}{2}ab = 60 \implies ab = 120, \\ &a + b + \sqrt{a^2 + b^2} = 40. \end{aligned}$$

Rearranging and squaring the latter,

$$\begin{aligned} &\sqrt{a^2 + b^2} = 40 - a - b \\ \Longrightarrow\ &a^2 + b^2 = 1600 + a^2 + b^2 - 80a - 80b + 2ab \\ \Longrightarrow\ &800 - 40a - 40b + ab = 0. \end{aligned}$$

Substituting $ab = 120$, we get $920 - 40a - 40b = 0$, so $23 = a + b$. Solving this simultaneously with $ab = 120$ gives the side lengths as $(8, 15, 17)$.

2993. The graphs $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ intersect at $x = k$. Furthermore, they have the same gradient at this point, meaning that they are tangential. So, the equation $\mathrm{f}(x) = \mathrm{g}(x)$ must have a double root at $x = k$. And the equation $\mathrm{f}(x) = \mathrm{g}(x)$ is at most a cubic, which leaves only one possible root elsewhere. Hence, $\mathrm{f}(x) = \mathrm{g}(x)$ has at most two distinct roots. □

2994. (a) The denominator is zero at $x = 5$.

(b) Expressing the improper fraction properly:

$$\frac{5 - 2x - 4x^2}{5 - x} \equiv 4x + 22 + \frac{105}{x - 5}.$$

As $x \to \pm\infty$, the fraction tends to zero, leaving an oblique asymptote at $y = 4x + 22$.

— Alternative Method —

The polynomial long division in the above is

$$\begin{array}{rrrr} & 4x & +22 & \\ \hline -x+5) & -4x^2 & -2x & +5 \\ & 4x^2 & -20x & \\ \hline & & -22x & +5 \\ & & 22x & -110 \\ \hline & & & -105 \end{array}$$

2995. (a) Looking for intersections,

$$\begin{aligned} &(x + 1/2)^2 - x - c^2 - \tfrac{1}{4} = 0 \\ \Longrightarrow\ &x^2 + x + \tfrac{1}{4} - x - c^2 - \tfrac{1}{4} = 0 \\ \Longrightarrow\ &x^2 - c^2 = 0 \\ \Longrightarrow\ &x = \pm c. \end{aligned}$$

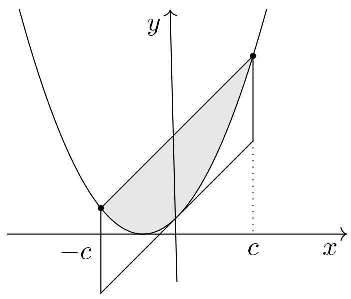
So, $x_1 = -c$ and $x_2 = c$. Hence, $x_2 - x_1 = 2c$, as required.

(b) The distance h, in the y direction, between the parabola and the line is given by

$$\begin{aligned} h &= x + c^2 + \tfrac{1}{4} - \left(x + \tfrac{1}{2}\right)^2 \\ &\equiv c^2 - x^2. \end{aligned}$$

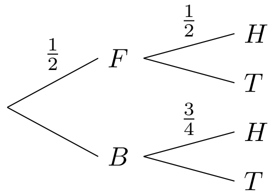
So, the vertical distance between the curve and the line is maximised at the y axis, at $h = c^2$.

(c) Using (b), the shaded region is smaller than the parallelogram shown:



This parallelogram has area $2c \times c^2$. So, the area of the shaded region satisfies $A < 2c^3$.

2996. Conditioning on the choice of coin, the possibility space is



Restricting the possibility space to heads,

$$\mathbb{P}(\text{biased} \mid \text{heads}) = \frac{\frac{1}{2} \times \frac{3}{4}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{3}{4}} = \frac{3}{5}.$$

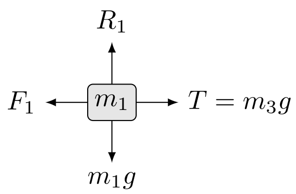
2997. Using log rules,

$$\begin{aligned} &\log_{p^2}\left(p^n\right) \times \log_{p^n} p \\ \equiv\ & n \log_{p^2} p \times \log_{p^n} p \\ \equiv\ & n \times \tfrac{1}{2} \times \tfrac{1}{n} \\ \equiv\ & \tfrac{1}{2}. \end{aligned}$$

— Nota Bene —

The evaluations from the second line to the third are done by definition. What do you have to raise p^2 by to get p? Answer: $1/2$. What do you have to raise p^n by to get p? Answer: $1/n$.

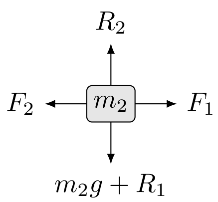
2998. Assuming (limiting) equilibrium, the tension in the string is $m_3 g$. For the upper block, the forces are:



Horizontally, F_1 is also $m_3 g$. The reaction between the stacked blocks is $R_1 = m_1 g$. Since we are in limiting equilibrium, this gives

$$\begin{aligned} m_3 g &= \mu_1 m_1 g \\ \Longrightarrow \mu_1 &= \frac{m_3}{m_1}. \end{aligned}$$

The forces on the lower block are:



Vertically, $R_2 = m_1 g + m_2 g$. And the frictional force F_1 rightwards is $m_3 g$. So, F_2 is also $m_3 g$. For limiting equilibrium,

$$\begin{aligned} m_3 g &= \mu_2 (m_1 g + m_2 g) \\ \Longrightarrow \mu_2 &= \frac{m_3}{m_1 + m_2}. \end{aligned}$$

Collating the results, both frictions are limiting if

$$\mu_1 = \frac{m_3}{m_1}, \quad \mu_2 = \frac{m_3}{m_1 + m_2}.$$

2999. For partial fractions, we require

$$\begin{aligned} \frac{3x+3}{x^2+3x} &\equiv \frac{A}{x+3} + \frac{B}{x} \\ \Longrightarrow 3x + 3 &\equiv Ax + B(x+3). \end{aligned}$$

Setting $x = 0, -3$ gives $B = 1$ and $A = 2$. So,

$$\frac{3x+3}{x^2+3x} \equiv \frac{2}{x+3} + \frac{1}{x}.$$

We can now integrate:

$$\begin{aligned} &\int \frac{2}{x+3} + \frac{1}{x}\, dx \\ \equiv\ & 2\ln|x+3| + \ln|x| + c \\ \equiv\ & \ln(x+3)^2 + \ln|x| + c \\ \equiv\ & \ln\left|x(x+3)^2\right| + c \\ \equiv\ & \ln\left|x^3 + 6x^2 + 9x\right| + c, \text{ as required.} \end{aligned}$$

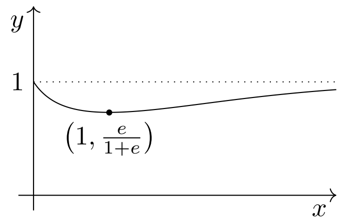
3000. The numerator is never zero, so the curve has no x axis intercepts. It crosses the y axis at $(0, 1)$.

By the quotient rule, the first derivative is

$$\frac{dy}{dx} = \frac{e^x(x-1)}{(x+e^x)^2}.$$

Setting the numerator to zero for SPs gives $x = 1$, where $y = \frac{e}{1+e} \approx 0.73$.

As $x \to \infty$, x becomes negligible compared to e^x, so the curve tends asymptotically to $y = 1$.



— End of Volume III —